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Flows associated with irregular $\mathbb{R}^d$—vector fields☆

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Abstract

This work consists on the study of flows associated with non-smooth $\mathbb{R}^d$—vector fields, namely concerning existence and uniqueness for almost—every initial condition. It is also proved that the flows avoid some special compact sets.

Keywords: Ordinary differential equations; Non-smooth vector fields

1. Introduction

We consider in this work ordinary differential equations on $\mathbb{R}^d$ associated to non-smooth vector fields. Instead we assume that the vector fields together with their gradient (in the distributions sense) and the exponential of the divergence satisfy $L^p(\sigma)$-type hypothesis. Here, the measure σ denotes the standard Gaussian measure on $\mathbb{R}^d$.

Such problems have been studied, in particular, by Di Perna and Lyons [DiP-L]. Because motivations come largely from Fluid Mechanics, the divergence (with respect to Lebesgue measure) of the vector field was first assumed to be zero, but the techniques have been subsequently generalized, for instance in [D]. These results rely on the

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analysis of the associated (partial differential) transport equations, $\frac{\partial u}{\partial t} = B \cdot \nabla u$, where B is the vector field.

On the other hand, the divergence accounts for the infinitesimal action of the flow on the measure space, and therefore, under suitable integrability conditions, can be integrated in order to obtain the density of the flow. These arguments were used in [C], in order to prove non-explosion of solutions of ordinary differential equations, almost - everywhere with respect to the initial conditions.

In this work, we use partly the techniques of DiPerna and Lions [DiP-L] based on the study of the transport equations and partly the more probabilistic arguments based on the study of the action of the flow on the Gaussian measure. We obtain existence and uniqueness of solutions as well as an expression for their density (a.e.) continuity of the trajectories and the flow property.

Finally, in the last paragraph, we generalize for our flows the non-avoidance of sets property discussed in [A].

2. Notations and main results

Let σ be the Gaussian measure on $\mathbb{R}^d$,

$$d\sigma(x) = (2\pi)^{-\frac{d}{2}} \mathrm{e}^{-\frac{1}{2}|x|^2}\, dx. \tag{2.1}$$

We shall denote by $W^{\alpha,p}(\sigma)$ the Sobolev spaces with respect to the Gaussian measure σ, namely

$$W^{\alpha,p}(\sigma) = \left\{ \varphi : \int_{\mathbb{R}^d} |\varphi|^p \, d\sigma < +\infty,\ \int_{\mathbb{R}^d} |\partial^k \varphi|^p \, d\sigma < +\infty \text{ for } |k| \leqslant \alpha \right\},$$

where $k = (k_1, \ldots, k_d)$, $|k| = k_1 + \cdots + k_d$, the partial derivatives $\partial^k \varphi = \frac{\partial^{|k|}\varphi}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}}$ are defined in the distribution sense and by $W^{\alpha,p}(dx)$ the Sobolev spaces with respect to the Lebesgue measure *dx*. Analogous notations $L^p(\sigma)$, $L^p(dx)$ will be considered with respect to integrability. We shall also consider

$$W^{\alpha,p}_{\mathrm{loc}}(\sigma) = \left\{ \varphi : \int_K |\varphi|^p \, d\sigma < +\infty, \int_K |\partial^k \varphi|^p \, d\sigma < +\infty, |k| \leqslant \alpha, \right. \\ \left. \forall \text{ compact } K \subset \mathbb{R}^d \right\}.$$

We represent by δ_σ the divergence operator associated with the measure σ. This operator corresponds to the adjoint of the gradient in $L^2(\sigma)$. Given a vector field B on $\mathbb{R}^d$, $\delta_\sigma B$

is defined as follows:

$$\int_{\mathbb{R}^d} B(x) \cdot \nabla \varphi(x)\, d\sigma(x) = \int_{\mathbb{R}^d} \delta_\sigma B(x) \varphi(x)\, d\sigma(x) \quad \forall \varphi \in W^{1,2}(\sigma).$$

If B is time-dependent, for fixed t we denote $B(t)(x) = B(t,x)$ and define $\delta_\sigma B(t)$ accordingly.

The divergence operator δ_σ is related with the divergence operator associated with the Lebesgue measure, usually denoted by div, by

$$\delta_\sigma B_t(x) = -\mathrm{div}\, B_t + x \cdot B_t(x). \tag{2.2}$$

To regularize the vector field we use the Ornstein–Uhlenbeck semigroup P_t, which is defined by

$$P_t\, f(x) = \int_{\mathbb{R}^d} f(\mathrm{e}^{-t} x + \sqrt{1-\mathrm{e}^{-2t}}\, y)\, d\sigma(y). \tag{2.3}$$

Among the properties of the Ornstein–Uhlenbeck semigroup, we have

$$\| P_t\, f \|_{L^p(\sigma)} \leqslant \| f \|_{L^p(\sigma)} \tag{2.4}$$

and a commutation relation between the semigroup operator and the divergence operator

$$\delta_\sigma (P_t B) = \mathrm{e}^t P_t (\delta_\sigma B). \tag{2.5}$$

We shall prove the following:

Theorem 2.1. *Let* $B = B(t,x)$ *be a time-dependent* $\mathbb{R}^d$*-valued vector field satisfying the following conditions*:
(1) $B \in L^1([0,T]; W^{1,1}_{\mathrm{loc}}(\sigma))$,
(2) $B \in L^1([0,T]; L^{1+\varepsilon}(\sigma))$, $\varepsilon > 0$,
(3) $B \in L^1([0,T]; L^1(dx))$,
(4) $\exists\ \lambda > 0$:

$$\int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{\lambda(|\delta_\sigma B|)}\, d\sigma(x)\, dt < +\infty. \tag{2.6}$$

Then there exist functions $U_{s,t}$ *and* $\tilde{U}_{s,t}$ σ*-a.e. defined for* $0 \leqslant s \leqslant t \leqslant T$ *such that*

$$U_{s,t}(x) = x + \int_s^t B(\tau, U_{s,\tau}(x))\, d\tau \quad \text{a.e. for} \quad s \leqslant t \tag{2.7}$$

and

$$\tilde{U}_{s,t}(x) = x - \int_s^t B(\tau, \tilde{U}_{\tau,t}(x))\, d\tau \quad \text{a.e. for} \quad s \leqslant t. \tag{2.8}$$

The trajectories $t \to U_{s,t}$ *defined on* $[s, T]$ *and* $t \to \tilde{U}_{s,t}$ *defined on* $[0, t]$ *are a.e. continuous.*

Moreover, if we denote $K_{s,t} = d(U_{s,t})_* \sigma / d\sigma$ *and* $\tilde{K}_{s,t} = d(\tilde{U}_{s,t})_* \sigma / d\sigma$, *we have*

$$K_{s,t} = \exp\left(\int_s^t \delta_\sigma B_r(\tilde{U}_{r,t})\, dr \right) \tag{2.9}$$

and

$$\tilde{K}_{s,t} = \exp\left(- \int_s^t \delta_\sigma B_r(U_{s,r})\, dr \right). \tag{2.10}$$

We also have the flow property, more precisely

$$U_{s,t} = U_{\tau,t} \circ U_{s,\tau} \quad a.e., \tag{2.11}$$

$$\tilde{U}_{s,t} = \tilde{U}_{s,\tau} \circ \tilde{U}_{\tau,t} \quad a.e. \tag{2.12}$$

for $0 \leqslant s \leqslant \tau \leqslant t \leqslant T$. *The flow* $\tilde{U}_{s,\tau}$ *is the inverse of* $U_{s,t}$, *in the sense that*

$$U_{s,t} \circ \tilde{U}_{s,t}(x) = \tilde{U}_{s,t} \circ \tilde{U}_{s,t}(x) = x \quad a.e. \tag{2.13}$$

for $0 \leqslant s \leqslant t \leqslant T$. *The flow satisfying these properties is (a.e.) unique.*

The following result can also be established:

Theorem 2.2. *Let* $B = B(t, x)$ *be a time-dependent* $\mathbb{R}^d$*-valued vector field satisfying the following conditions:*
(1) $B \in L^1([0, T]; W^{1,1}_{\text{loc}}(\sigma))$;
(2) $\exists\ \lambda > 0$:

$$\int_0^T \int_{\mathbb{R}^d} e^{\lambda(|\delta_\sigma B| + |B|)}\, d\sigma(x)\, dt < +\infty. \tag{2.14}$$

Then there exist functions $U_{s,t}$ *and* $\tilde{U}_{s,t}$ *a.e. defined for* $0 \leqslant s \leqslant t \leqslant T$ *which satisfies all properties of the Theorem* 2.1.

3. The existence of the flow (Theorem 2.1)

We first regularize the vector field B. Let α be a smooth positive function on $\mathbb{R}$ with compact support and such that $\int_{\mathbb{R}} \alpha(x)\,dx = 1$. We consider $\alpha_n(t) = n\alpha(nt)$ and $B^{(n)}(t,x) = (B(\cdot,x) * \alpha_n(\cdot))(t)$. We define

$$B^n(t,x) = P_{\frac{1}{n}} B^{(n)}(t,x). \tag{3.1}$$

It follows from the properties of the semigroup $\mathcal{P}_t$ that the vector field B^n is smooth, therefore it defines a smooth local flow

$$U^n_{s,t}(x) = x + \int_s^t B^n(\tau, U^n_{s,\tau}(x))\,d\tau \quad \forall x \in \mathbb{R}^d. \tag{3.2}$$

We first consider the assumptions of Theorem 2.1. Since

$$\|B^n\|_{L^\infty([0,T]\times\mathbb{R}^d)} \leqslant \|B\|_{L^1([0,T];L^1(dx))}, \tag{3.3}$$

the flow is defined for all $t \in [s,T]$.

For every $s \leqslant t$, the map $U^n_{s,t}(\cdot)$ is a diffeomorphism on $\mathbb{R}^d$ and the inverse $\tilde{U}^n_{s,t}(\cdot)$ satisfies

$$\tilde{U}^n_{s,t}(x) = x - \int_s^t B^n(\tau, \tilde{U}^n_{\tau,t}(x))\,d\tau \quad \forall x \in \mathbb{R}^d. \tag{3.4}$$

The flow property

$$U^n_{s,t}(x) = U^n_{\tau,t} \circ U^n_{s,\tau}(x), \quad s \leqslant \tau \leqslant t, \tag{3.5}$$

$$\tilde{U}^n_{s,t}(x) = \tilde{U}^n_{s,\tau} \circ \tilde{U}^n_{\tau,t}(x), \quad s \leqslant \tau \leqslant t \tag{3.6}$$

holds for all $x \in \mathbb{R}^d$.

The measures $(U^n_{s,t})_*\sigma$ and $(\tilde{U}^n_{s,t})_*\sigma$ are absolutely continuous with respect to the Gaussian measure σ and the density functions, denoted by $K^n_{s,t}$ and $\tilde{K}^n_{s,t}$ respectively, can be written as

$$K^n_{s,t}(x) = \exp\left(\int_s^t \delta_\sigma B^n_r(\tilde{U}^n_{r,t})\,dr\right) \tag{3.7}$$

and

$$\tilde{K}^n_{s,t}(x) = \exp\left(-\int_s^t \delta_\sigma B^n_r(U^n_{s,r})\,dr\right). \tag{3.8}$$

In order to prove the existence of the flow in Theorem 2.1, we follow the method of DiPerna and Lions [DiP-L]. We consider the forward and backward transport equations which correspond respectively to

$$\frac{\partial}{\partial t}u - B \cdot \nabla u = 0 \tag{3.9}$$

with an initial condition and

$$\frac{\partial}{\partial t}u + B \cdot \nabla u = 0 \tag{3.10}$$

with a final condition.

In fact it is enough to consider just one of these equations, since their solutions are related by change of variables.

Differentiating equality (3.5) with respect to the variable s at $\tau = s$, we can verify that $U(s,x) = U^n_{s,t}(x)$ solves the vectorial backward transport equation (3.10), where B is replaced by B^n, with final condition $U(t,x) = U^n_{t,t}(x) = x$.

We start with a result concerning the uniform estimation of the $L^p(\sigma)$ norms of the density functions.

Lemma 3.1. *Under the assumptions of Theorem* 1.1 *and for $t-s$ small enough there exists a constant C which depends only on p and on the value of the integral in* (2.6) *such that* $\|K^n_{s,t}\|_{L^p(\sigma)} \leqslant C$.

Proof. Using the expression (3.7) (cf. [C,U-Z]), we have

$$\int_{\mathbb{R}^d} |K^n_{s,t}(x)|^p \, d\sigma(x) \leqslant \frac{\mathrm{e}^{\frac{t-s}{p\varepsilon}}}{q\varepsilon} \int_{\mathbb{R}^d} \int_s^t \mathrm{e}^{p[(t-s)\vee\varepsilon]\delta_\sigma B^n_r(x)} \, dr \, d\sigma(x) \tag{3.11}$$

with $0 \leqslant s \leqslant t$, $\varepsilon > 0$, $\frac{1}{p} + \frac{1}{q} = 1$. If $\varepsilon = \frac{\lambda}{p\mathrm{e}}$ and $t - s < \varepsilon$, we obtain

$$\int_{\mathbb{R}^d} |K^n_{s,t}(x)|^p \, d\sigma(x) \leqslant c(\lambda, p) \int_{\mathbb{R}^d} \int_0^T \mathrm{e}^{\frac{\lambda}{\mathrm{e}}|\delta_\sigma B^n_r(x)|} \, dr \, d\sigma(x). \tag{3.12}$$

From properties (2.4) and (2.5) of the Ornstein–Uhlenbeck semigroup we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathrm{e}^{\frac{\lambda}{\mathrm{e}}|\delta_\sigma B^n_r(x)|} \, d\sigma(x) &= \sum_k \frac{\lambda^k}{\mathrm{e}^k k!} \int_{\mathbb{R}^d} |\delta_\sigma B^n_r(x)|^k \, d\sigma(x) \\
&\leqslant \sum_k \frac{\lambda^k}{k!} \int_{\mathbb{R}^d} |\delta_\sigma B^{(n)}_r(x)|^k \, d\sigma(x) \\
&= \int_{\mathbb{R}^d} \mathrm{e}^{\lambda|\delta_\sigma B^{(n)}_r(x)|} \, d\sigma(x).
\end{aligned}$$

Since

$$\int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{\lambda|\delta_\sigma B_r^{(n)}(x)|}\, d\sigma(x)\, dr \leqslant \int_0^T \int_{\mathbb{R}^d} \mathrm{e}^{\lambda|\delta_\sigma B_r(x)|}\, d\sigma(x)\, dr + T,$$

the result follows. □

In the next two results we study the solutions of the transport equations, these solutions being considered in the distribution sense with respect to the Gaussian measure σ. Analogous results were established in [DiP-L] in the context of the Lebesgue measure. In particular, the first lemma enables us to approximate any $L^1([0,T]; L^\infty(\sigma))$ solution of the transport equation by smooth solutions of approximate equations, under the regularity assumption $L^1([0,T]; W^{1,1}_{\mathrm{loc}}(\sigma))$ for the vector field B. We regularize the solution u of the transport equation, as in [DiP-L], by convolution with ρ_ε where $\rho_\varepsilon(x) = \frac{1}{\varepsilon^N}\rho(\frac{x}{\varepsilon})$, ρ is C^∞ with compact support and $\int_{\mathbb{R}^d} \rho\, dx = 1$.

Lemma 3.2. *Let $B \in L^1([0,T]; W^{1,\alpha}_{\mathrm{loc}}(\sigma))$ and let $u \in L^\infty([0,T]; L^p_{\mathrm{loc}}(\sigma))$ be a solution in the distributions sense of Eq.* (3.9). *Then, the function u_n defined by $u_n = u * \rho_{\frac{1}{n}}$ satisfies the following partial differential equation*:

$$\begin{aligned} &\frac{\partial}{\partial t} u_n(t,x) = B(t,x)\cdot \nabla u_n(t,x) + r_n(t,x), \ t > s, \\ &u_n(s,x) = u_s(x) \qquad\qquad \forall\, x \in \mathbb{R}^d, \end{aligned} \tag{3.13}$$

where $r_n \to 0$ in $L^1([0,T]; L^\beta_{\mathrm{loc}}(\sigma))$ with $\frac{1}{\alpha} + \frac{1}{p} = \frac{1}{\beta}$.

Proof. Since $B \in L^1([0,T]; W^{1,\alpha}_{\mathrm{loc}}(\sigma))$ and $u \in L^\infty([0,T]; L^p_{\mathrm{loc}}(\sigma))$, we have $B \in L^1([0,T]; W^{1,\alpha}_{\mathrm{loc}}(dx))$ and $u \in L^1([0,T]; L^p_{\mathrm{loc}}(dx))$ and we can apply the result of DiPerna and Lions [DiP-L]. □

As a consequence of this result, for every function β in $C^1(\mathbb{R})$ with bounded derivative β', $\beta(u_n)$ satisfies

$$\frac{\partial \beta(u_n)}{\partial t} - B\cdot \nabla \beta(u_n) = r_n \beta'(u_n). \tag{3.14}$$

Taking the limit as $n \to \infty$ we obtain

$$\frac{\partial \beta(u)}{\partial t} - B\cdot \nabla \beta(u) = 0. \tag{3.15}$$

Lemma 3.3. *Let B satisfy assumptions* (1), (2) *and* (4) *of the Theorem* 2.1. *For T small enough, Eq.* (3.9) *with initial condition $u_0(s)$ has at most one solution in* $L^\infty([0,T]; L^\infty(\sigma)))$.

Proof. Let u_1 and u_2 be bounded solutions in distributions sense of Eq. (3.9) with the same initial condition $u_0(s)$. We consider $u = u_1 - u_2$. Since the equation is linear, u is also a solution of (3.9) with initial condition $u(s) = 0$. Let $M > 0$ and consider $\beta(\xi) = |\xi| \wedge M$. Approximating β by functions in $C^1(\mathbb{R})$ we obtain

$$\frac{\partial \beta(u)}{\partial t} = B \cdot \nabla \beta(u). \tag{3.16}$$

Let ψ be a positive smooth function defined on $\mathbb{R}^d$ such that $\psi(x) = 1$, if $|x| \leqslant 1$ and $\psi(x) = 0$, if $|x| > 2$. For $N > 1$, we consider the functions $\psi_N(x) = \psi(\frac{x}{N})$. Multiplying the equation by ψ_N and integrating with respect to the measure σ we obtain

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^d} \beta(u)\psi_N \, d\sigma &= \int_{\mathbb{R}^d} B(t)\cdot \nabla\beta(u)\psi_N \, d\sigma \\
&= -\int_{\mathbb{R}^d} \beta(u)\delta_\sigma(B\cdot\psi_N)\, d\sigma \\
&= -\int_{\mathbb{R}^d} \beta(u)\psi_N\delta_\sigma B(t)\, d\sigma + \frac{1}{N}\int_{\mathbb{R}^d}\beta(u) B\cdot\nabla\psi \, d\sigma.
\end{aligned}$$

We have

$$\frac{1}{N}\int_{\mathbb{R}^d}\beta(u)B\cdot\nabla\psi\, d\sigma \leqslant \frac{C}{N}\int_{N\leqslant|x|\leqslant 2N}|B|\, d\sigma.$$

From the hypothesis (2) of Theorem 2.1, this integral converges to zero as $N\to\infty$. Taking the limit when $N\to\infty$ we obtain

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\beta(u)\, d\sigma = -\int_{\mathbb{R}^d}\beta(u)\delta_\sigma B(t)\, d\sigma. \tag{3.17}$$

Let $\Omega_k(t)$ be the subset of $\mathbb{R}^d$ where $|\delta_\sigma B(t)| < K$.

We have

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{\mathbb{R}^d}\beta(u)\, d\sigma &\leqslant \int_{\Omega_k(t)}\beta(u)|\delta_\sigma B(t)|\, d\sigma + \int_{\Omega_k^c(t)}\beta(u)|\delta_\sigma B(t)|\, d\sigma \\
&\leqslant K\int_{\mathbb{R}^d}\beta(u)\, d\sigma + \int_{\Omega_k^c(t)}\beta(u)|\delta_\sigma B(t)|\, d\sigma
\end{aligned}$$

that implies

$$\int_{\mathbb{R}^d} \beta(u)\, d\sigma \leqslant \int_s^t \left(K \int_{\mathbb{R}^d} \beta(u)\, d\sigma \right) d\tau + \int_s^T \int_{\Omega_k^c(\tau)} \beta(u) |\delta_\sigma B|\, d\sigma\, d\tau.$$

Using Gronwall's inequality, we deduce

$$\int_{\mathbb{R}^d} \beta(u) d\sigma \leqslant \mathrm{e}^{(t-s)K} \left(\int_s^T \int_{\Omega_k^c(\tau)} \beta(u) |\delta_\sigma B(\tau)|\, d\sigma\, d\tau \right).$$

We have

$$\begin{aligned} \sigma(\Omega_k^c(t)) &= \sigma(\{x : |\delta_\sigma B(t)| > K\}) \\ &= \sigma(\{x : \mathrm{e}^{\lambda |\delta_\sigma B(t)|} > \mathrm{e}^{\lambda K}\}) \\ &\leqslant \mathrm{e}^{-\lambda K} \int_{\mathbb{R}^d} \mathrm{e}^{\lambda |\delta_\sigma B(t)|}\, d\sigma. \end{aligned}$$

Using the assumption (2.6) we obtain

$$\int_s^T \sigma(\Omega_k^c(\tau)) \leqslant C \mathrm{e}^{-\lambda K},$$

where C is a constant.

Therefore

$$\begin{aligned} \int_s^T \int_{\Omega_k^c(\tau)} \beta(u) |\delta_\sigma B(\tau)|\, d\sigma\, d\tau &\leqslant \left(\int_s^T \int_{\mathbb{R}^d} |\delta_\sigma B(\tau)|^2\, d\sigma d\tau \right)^{\frac{1}{2}} \left(\int_s^T \sigma(\Omega_k^c(\tau))\, d\tau \right)^{\frac{1}{2}} \\ &\leqslant C \mathrm{e}^{-\frac{\lambda K}{2}}. \end{aligned}$$

If we consider $t - s$ small enough, the result follows taking the limit in K when $K \to \infty$. $\square$

In the following, given a sequence ξ_n of functions defined on some measurable space (X, μ) with values in a Banach space M (endowed with the norm $\|\cdot\|$), we say that ξ_n converges to ξ in $L^0(\mu; M)$ if for all $\varepsilon > 0$, $\mu\{\|\xi_n - \xi\| > \varepsilon\} \to 0$.

Lemma 3.4. *Let B satisfy the assumptions of the Theorem* 1.1. *For* $s, t \in (0, T)$ *with* $t - s$ *small enough, let us denote by dt the Lebesgue measure on the interval* $[s, t]$.

We have

(a) *The sequence* $U^n_{s,r}(\cdot)$ *converges in* $L^0(\sigma;\mathbb{R}^d)$.

(b) *The sequence* $U^n_{s,\cdot}(\cdot)$ *converges in* $L^0(dt\times\sigma;\mathbb{R}^d)$ (*the limit will be denoted by* $U_{s,t}$).

Proof. Let β be a continuous and bounded function on $\mathbb{R}$. Denote by $U^{n(i)}_{s,t}$ the i-component of $U^n_{s,t}$. The sequences $v^i_n=\beta(U^{n(i)}_{s,t})$ and $w^i_n=\beta^2(U^{n(i)}_{s,t})$ are bounded sequences on $L^\infty([0,T]\times[0,T];L^\infty(\mathbb{R}^d))$, so there exists subsequences which converge in $L^\infty((0,T)^2\times\mathbb{R}^d)$ - * to v^i and w^i, respectively. The functions v^i and w^i are bounded solutions of the transport equation with initial conditions $\beta(x_i)$ and $\beta^2(x_i)$, respectively. On the other hand, as written after Lemma 3.2 the function $(v^i)^2$ also satisfies the transport equation with initial condition $\beta^2(x_i)$. From the uniqueness result, $w^i=(v^i)^2$. Therefore, up to a subsequence, $(v^i_n)^2$ converges to $(v^i)^2$ in $L^\infty((0,T)^2\times\mathbb{R}^d)$ - * which implies $v^i_n\to v^i$ in $L^2((0,T)^2\times\mathbb{R}^d)$ (and $L^2([0,T]\times[0,T];L^2_{\mathrm{loc}}(\sigma))$), consequently we have convergence in measure $dt\times\sigma$ (and σ). Since $\beta(U^{n(i)}_{s,t})$ converges in measure to some function v, and β with the above regularity is arbitrary, $U^{n(i)}_{s,\cdot}(\cdot)$ should converge in measure $dt\times\sigma$ to some function $U^{(i)}_{s,\cdot}(\cdot)$ (and $U^{n(i)}_{s,t}(\cdot)$ should converge in measure σ to some function $U^{(i)}_{s,t}(\cdot)$ and $v(\cdot,\cdot)=\beta(U^{(i)}_{s,\cdot}(\cdot))$. $\square$

The measurable function $U_{s,t}$ keeps the measure σ quasi-invariant.

Lemma 3.5. *Let* $U_{s,t}$ *be the function defined in above lemma with* $s,t\in(0,T)$ *and let* T *be small enough. The measure* $(U_{s,t})_*\sigma$ *is absolutely continuous with respect to the measure* σ, *and the corresponding density* $K_{s,t}$ *belongs in* $L^p(\sigma)$ *for all* $p>1$.

Proof. Since $U^n_{s,t}$ converges to $U_{s,t}$ in measure, for every function f continuous with compact support

$$\int_{\mathbb{R}^d} f(U^n_{s,t})\,d\sigma\to\int_{\mathbb{R}^d} f(U_{s,t})\,d\sigma. \tag{3.18}$$

On the other hand, by Lemma 3.1, there exists a subsequence $K^{n_k}_{s,t}$ of $K^n_{s,t}$ which converges weakly in $L^p(\sigma)$ to some function $K_{s,t}$. Therefore

$$\int_{\mathbb{R}^d} f(U^{n_k}_{s,t})\,d\sigma=\int_{\mathbb{R}^d} f(x)K^{n_k}_{s,t}(x)\,d\sigma \tag{3.19}$$

converges to

$$\int_{\mathbb{R}^d} f(x)K_{s,t}(x)\,d\sigma \tag{3.20}$$

and the result follows. $\square$

The next convergence result will be useful to pass to the limit the integral equation and to deduce some properties for $U_{s,t}(x)$ from the corresponding one for $U^n_{s,t}(x)$ (cf. [U-Z]).

Lemma 3.6. *We consider the hypothesis of Theorem* 2.1. *Let M be any Banach space and F any M-valued random variable. Let F_1 be a $\mathbb{R}^d$-valued random variable and $s < t < T$ with $t - s$ small enough. Then we have*

$$\begin{array}{ll}\text{(i)}\ F(U^n_{s,t}(\cdot)) \to F(U_{s,t}(\cdot)) & \textit{in } L^0(\sigma; M)\\ \text{(ii)}\ F_1(\cdot, U^n_{s,\cdot}(\cdot)) \to F_1(\cdot, U_{s,\cdot}(\cdot)) & \textit{in } L^0(dt \times \sigma; \mathbb{R}^d),\end{array}$$

where dt is the Lebesgue measure on $[s, t]$.

Proof. We prove the first statement. The proof of the second one is similar.

Let us suppose that $F \in L^\infty(\sigma)$ and denote by $\|\cdot\|$ the norm on M. From Lemma 3.4 $U^n_{s,t}$ converges to $U_{s,t}$ in measure-σ. If necessary, by taking a subsequence, we can consider that $U^n_{s,t}$ converges to $U_{s,t}$ σ-a.e.

For all $\varepsilon > 0$ there exists $K_\varepsilon \subset \mathbb{R}^d$ such that $\sigma(K^c_\varepsilon) < \varepsilon$ and $F|_{K_\varepsilon}$ is uniformly continuous. Fixed s, t,

$$\begin{aligned}\int_{\mathbb{R}^d} \|F(U^n_{s,t}) - F(U_{s,t})\| &= \int_{\{U^n_{s,t}, U_{s,t} \in K_\varepsilon\}} \|F(U^n_{s,t}) - F(U_{s,t})\|\\ &\quad + \int_{\mathbb{R}^d - \{U^n_{s,t}, U_{s,t} \in K_\varepsilon\}} \|F(U^n_{s,t}) - F(U_{s,t})\| = I_1(n) + I_2(n).\end{aligned}$$

By continuity, given $\delta > 0$ there exists a constant c such that

$$\begin{aligned}I_1(n) &= \int_{\{U^n_{s,t}, U_{s,t} \in K_\varepsilon | U^n_{s,t} - U_{s,t}| < c\}} \|F(U^n_{s,t}) - F(U_{s,t})\|\\ &\quad + \int_{\{U^n_{s,t}, U_{s,t} \in K_\varepsilon | U^n_{s,t} - U_{s,t}| > c\}} \|F(U^n_{s,t}) - F(U_{s,t})\|\\ &\leqslant \delta + \left(\int_{\mathbb{R}^d} \|F(U^n_{s,t}) - F(U_{s,t})\|^2\right)^{\frac{1}{2}} (\sigma\{|U^n_{s,t} - U_{s,t}| > c\})^{\frac{1}{2}},\\ I_2(n) &= \int_{\{U^n_{s,t} \in K_\varepsilon, U_{s,t} \in K^c_\varepsilon\}} \|F(U^n_{s,t}) - F(U_{s,t})\|\\ &\quad + \int_{\{U^n_{s,t} \in K^c_\varepsilon, U_{s,t} \in K_\varepsilon\}} \|F(U^n_{s,t}) - F(U_{s,t})\|\end{aligned}$$

$$+\int_{\{U^n_{s,t}\in K^c_\varepsilon, U_{s,t}\in K^c_\varepsilon\}} \|F(U^n_{s,t}) - F(U_{s,t})\| = I_{2,1}(n) + I_{2,2}(n) + I_{2,3}(n),$$

$$I_{2,1}(n) \leqslant C_1 \left(\int_{\mathbb{R}^d} \|F(U^n_{s,t}) - F(U_{s,t})\|^2\right)^{\frac{1}{2}} (\sigma(K^c_\varepsilon))^{\frac{1}{2}},$$

where C_1 is a constant.

The analysis of $I_{2,2}(n), I_{2,3}(n)$ are similar to $I_{2,1}(n)$. If F is not in L^∞, we can consider $F_a \in L^\infty$, such that $\lim_{a\to 0} F_a(x) = F(x)$, σ- a.e-x. Then

$$\begin{aligned}\sigma\{\|F(U^n_{s,t}) - F(U_{s,t})\| > c\} \leqslant\ & \sigma\left\{\|F(U^n_{s,t}) - F_a(U^n_{s,t})\| > \frac{c}{3}\right\}\\ &+\mu\left\{\|F_a(U^n_{s,t}) - F_a(U_{s,t})\| > \frac{c}{3}\right\}\\ &+\mu\left\{\|F_a(U_{s,t}) - F(U_{s,t})\| > \frac{c}{3}\right\}.\end{aligned}$$

Using the uniform integrability of the densities we obtain the result. □

Therefore, considering the limit in measure of the integral equation as n goes to ∞, we obtain

Lemma 3.7. *Under the assumptions of the Theorem* 2.1, *the map* $U_{s,t}(x)$ *satisfies the integral equation*

$$U_{s,t}(x) = x + \int_s^t B(\tau, U_{s,\tau}(x))\, d\tau\, \sigma - a.e. \tag{3.21}$$

with $0\leqslant s < t$ *and* $t-s$ *small.*

Proof. We have

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_s^t |B^n(\tau, U^n_{s,\tau}(x)) - B(\tau, U_{s,\tau}(x))|\, ds\, d\sigma(x)\\ &\quad\leqslant \int_{\mathbb{R}^d}\int_s^t |B^n(\tau, U^n_{s,\tau}(x)) - B(\tau, U^n_{s,\tau}(x))|\, ds\, d\sigma(x)\\ &\qquad+\int_{\mathbb{R}^d}\int_s^t |B(\tau, U^n_{s,\tau}(x)) - B(\tau, U_{s,\tau}(x))|\, ds\, d\sigma(x)\\ &\quad\leqslant \int_s^t\int_{\mathbb{R}^d} |B^n(\tau, x) - B(\tau, x)| k^n_{s,\tau}\, ds\, d\sigma(x)\\ &\qquad+\int_{s^t}\int_{\mathbb{R}^d} |B(\tau, U^n_{s,\tau}(x)) - B(\tau, U_{s,\tau}(x))|\, ds\, d\sigma(x).\end{aligned}$$

By Lemma 3.1 and since B^n converges to B in $L^1([0,T];L^{1+\varepsilon}(\sigma))$ we deduce

$$\int_s^t \int_{\mathbb{R}^d} |B^n(\tau,x) - B(\tau,x)| k^n_{s,\tau}\, ds \leqslant C \int_0^T \left(\int_{\mathbb{R}^d} |B^n(\tau,x) - B(\tau,x)|^{1+\varepsilon} \right)^{\frac{1}{1+\varepsilon}} d\tau.$$

Taking $F_1 = B$ in Lemma 3.6 we obtain

$$B(\tau, U^n_{s,\tau}(x)) \to B(\tau, U_{s,\tau}(x))$$

with respect to the measure $dt \times \sigma$. Therefore there exists a subsequence which converges a.e.-$dt \times \sigma$. Moreover

$$\int_{[s,t]\times\mathbb{R}^d} |B(\tau, U^n_{s,\tau}(x))|^p \, d\sigma(x)\, dt(\tau) \leqslant \left(\int_{[s,t]\times\mathbb{R}^d} |B(\tau,x)|^{p^2} \, d\sigma(x)\, dt(\tau) \right)^{\frac{1}{p}} \times \left(\int_{[s,t]\times\mathbb{R}^d} K^q_{s,\tau}\, d\sigma(x)\, dt(\tau) \right)^{\frac{1}{q}}$$

with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p,\ p^2 < 1+\varepsilon$. Therefore

$$\sup_n \int_{[s,t]\times\mathbb{R}^d} |B(\tau, U^n_{s,\tau}(x))|^p \, d\sigma(x)\, dt(\tau) < \infty$$

and $B(\cdot, U^n_{s,\cdot}(\cdot))$ is absolutely integrable with respect to $dt \times \sigma$. Using the same notation for the subsequence, we obtain

$$\int_{[s,t]\times\mathbb{R}^d} |B(\tau, U^n_{s,\tau}(x)) - B(\tau, U_{s,\tau}(x))| \, ds \to 0$$

therefore

$$\int_s^t B^n(\tau, U^n_{s,\tau}))\, ds \to \int_s^t B(\tau, U_{s,\tau}))\, ds \quad \text{in } L^1(\sigma). \qquad \square \tag{3.22}$$

Given $T > 0$, we need some result in order to define the flow $U_{s,t}$ on the whole interval $[0,T]$.

Lemma 3.8. *Suppose $0 \leqslant t < \tau \leqslant T$ such that*

$$\int_t^\tau |B^n(r, U^n_{t,r}) - B^m(r, U^m_{t,r})| \, dr \to 0 \textit{ in measure } \sigma. \tag{3.23}$$

If $0 \leqslant s < t$ *where* $t - s$ *is small, then*

$$\int_s^\tau |B^n(r, U^n_{t,r}) - B^m(r, U^m_{t,r})|\, dr \to 0 \text{ in measure } \sigma.$$

Proof.

$$\int_s^\tau B^n(r, U^n_{s,r})\, dr = \int_s^t B^n(r, U^n_{s,r})\, dr + \int_t^\tau B^n(r, U^n_{s,r})\, dr. \tag{3.24}$$

Using the semigroup property for the regular flow $U^n_{s,t}$, we have

$$\int_t^\tau B^n(r, U^n_{s,r})\, dr = \int_t^\tau B^n(r, U^n_{t,r}(U^n_{s,t}))\, dr.$$

The convergence of the first term in the second member of (3.24) follows from (3.22). To prove the convergence of the second term, we consider the function $G_n(x)(r) = \chi_{[s,\tau]}(r) B^n(r, U^n_{t,r}(x)$. By (3.23) G_n converges to G in $L^0(\sigma; L^1([t,\tau]; \mathbb{R}^d))$. If we now apply Lemma 3.6 and use the uniform integrability of the densities we conclude that $G_n \circ U^n_{t,r}$ converges to $G \circ U_{t,r}$ in $L^0(\sigma; L^1([t,\tau]; \mathbb{R}^d))$ which is equivalent to write

$$\int_t^\tau |B^n(r, U^n_{t,r}(U^n_{s,t})) - B(r, U_{t,r}(U_{s,t}))|\, dr$$

converges to zero in measure σ. □

Using Lemma 3.8 a finite number of times, we obtain a subsequence $U^{n_k}_{s,t}$ such that

$$\int_s^T |B^{n_k}(r, U^{n_k}_{s,r}) - B^{n_l}(r, U^{n_l}_{s,r})|\, dr \tag{3.25}$$

converges to zero in measure σ as $k, l \to \infty$.

We can now complete the proof of Theorem 2.1. We will omit the proof of the existence of $\tilde{U}_{s,t}$ and their properties because we just need to repeat the arguments used for $U_{s,t}$.

We remark that (3.25) implies

$$\sup_{r \in [s,T]} |U^{n_k}_{s,r} - U^{n_l}_{s,r}| \to 0 \tag{3.26}$$

in measure σ, so $U_{s,t}$ is σ-a.e. defined for all $t \in [s, T]$, and is also continuous on the interval $[s, T]$.

From Lemmas 3.6–3.8 we conclude that for $0 \leqslant s < t \leqslant T$ there exists $U_{s,t}$, defined σ-a.e. such that

$$U_{s,t}(x) = x + \int_s^t B(\tau, U_{s,\tau}(x))\, d\tau \quad \text{for } s \leqslant t \leqslant T. \tag{3.27}$$

Now we prove that $U_{s,t}$ verifies the flow property (2.11). By Lemma 3.8

$$U^n_{\tau,t}(U^n_{s,\tau}) = U^n_{s,t} \to U_{s,t}$$

in measure and

$$|U^n_{\tau,t}(U^n_{s,\tau}) - U_{\tau,t}(U_{s,\tau})| \leqslant |U^n_{\tau,t}(U^n_{s,\tau}) - U_{\tau,t}(U^n_{s,\tau})| + |U_{\tau,t}(U^n_{s,\tau}) - U_{\tau,t}(U_{s,\tau})|.$$

If we consider $\tau - s$ small, from the uniform integrability of the density functions

$$|U^n_{\tau,t}(U^n_{s,\tau}) - U_{\tau,t}(U^n_{s,\tau})|$$

converges to zero in measure σ. Applying Lemma 3.6 with $F_1 = U_{\tau,t}$, we also have $|U_{\tau,t}(U^n_{s,\tau}) - U_{\tau,t}(U_{s,\tau})| \to 0$ in measure σ. If $\tau - s$ is not small there exists $m \in \mathbb{N}$ and r small such that $\tau = s + mr$,

$$\begin{aligned}
U_{\tau,t} \circ U_{s,\tau} &= U_{\tau,t} \circ (U_{s+r,\tau} \circ U_{s,s+r}) \\
&= U_{\tau,t} \circ (U_{s+2r,\tau} \circ U_{s+r,s+2r} \circ U_{s,s+r}) \\
&= \ldots \\
&= U_{\tau,t} \circ (U_{s+(m-1)r,\tau} \circ U_{s+(m-2)r,s+(m-1)r} \circ \cdots \circ U_{s+r,s+2r} \circ U_{s,s+r}) \\
&= (U_{\tau,t} \circ U_{s+(m-1)r,\tau}) \circ U_{s+(m-2)r,s+(m-1)r} \circ \cdots \circ U_{s+r,s+2r} \circ U_{s,s+r} \\
&= U_{s+(m-1)r,t} \circ U_{s+(m-2)r,s+(m-1)r} \circ \cdots \circ U_{s+r,s+2r} \circ U_{s,s+r} \\
&= \ldots \\
&= (U_{s+2r,t} \circ U_{s+r,s+2r}) \circ U_{s,s+r} \\
&= U_{s+r,t} \circ U_{s,s+r} \\
&= U_{s,t}.
\end{aligned}$$

The laws of the random variables $U_{s,t}$ are absolutely continuous with respect to σ, the density functions $K_{s,t}$ being given by the expressions

$$K_{s,t} = \exp\left(\int_s^t \delta_\sigma B_r(\tilde{U}_{r,t})\, dr \right). \tag{3.28}$$

In fact, from 3.6 it follows that there exists a subsequence of $\delta_\sigma B_r^n(\tilde{U}_{r,t}^n)$ which converges to $\delta_\sigma B_r(\tilde{U}_{r,t})$ in $L^0(dt\times\sigma;\mathbb{R}^d)$. Since $\delta_\sigma B_r^n(\tilde{U}_{r,t}^n)$ is uniformly integrable we have a subsequence which converges in $L^1(dt\times\sigma;\mathbb{R}^d)$. Therefore there exists a subsequence of $\int_s^t \delta_\sigma B_r^n(\tilde{U}_{r,t}^n)\,dr$ which converges to $\int_s^t \delta_\sigma B_r(\tilde{U}_{r,t})\,dr$ in $L^0(\sigma;\mathbb{R}^d)$. Since the sequence $K_{s,t}^n$ is uniformly integrable, we deduce that $K_{s,t}^n$ converges to $K_{s,t} = \exp(\int_s^t \delta_\sigma B_r(\tilde{U}_{r,t})\,dr)$ in $L^1(\sigma)$.

The next paragraph completes the proof of the Theorem 2.1.

4. Uniqueness of the flow

To prove that the flow associated with the vector field B is unique, we follow the strategy of DiPerna and Lions [DiP-L]. Let u_0 be an arbitrary function in $C^\infty(\mathbb{R}^d)$ with compact support. Let us define $v(s,x) = u_0(U_{s,t}(x))$, $s<t$. We shall prove that $v(s,x)$ is solution in the distributions sense of Eq. (3.10) with final condition $u_0(x)$.

Lemma 4.1. *The function $v(s,x)$ is solution in distributions sense of the partial differential equation* (3.10).

Proof. We consider

$$\begin{aligned}\Delta_h(s) &= \int_{\mathbb{R}^d} \frac{1}{h}[v(s+h,x)-v(s,x)]\varphi(x)\,d\sigma(x)\\ &= \int_{\mathbb{R}^d} \frac{1}{h}[u_0(U_{s+h,t}(x)) - u_0(U_{s,t}(x)]\varphi(x)\,d\sigma(x)\end{aligned}$$

for every φ in $C^\infty(\mathbb{R}^d)$ with compact support.

Using the flow property and knowing that $d(\tilde{U}_{s,t})_*\sigma = \tilde{K}_{s,t}\,d\sigma$ with $\tilde{K}_{s,t}$ given by (2.10) we have

$$\Delta_h(s) = \int_{\mathbb{R}^d} \frac{1}{h} u_0(U_{s,t}(x))[\varphi(U_{s,s+h}(x))\tilde{K}_{s,s+h} - \varphi(x)]\,d\sigma(x).$$

Since

$$\begin{aligned}\frac{\partial}{\partial h}[\varphi(U_{s,s+h}(x))\tilde{K}_{s,s+h}] = {} & \nabla\varphi(U_{s,s+h}(x))B(s+h,U_{s,s+h}(x))\tilde{K}_{s,s+h}(x)\\ & -\varphi(U_{s,s+h}(x))(\delta_\sigma B_{s+h})(U_{s,s+h}(x))\\ & \times\exp\left(-\int_s^{s+h}(\delta_\sigma B_r)(U_{s,r}(x))\,dr\right),\end{aligned}$$

we have

$$
\begin{aligned}
\Delta_h(s) = & \int_{\mathbb{R}^d} \frac{1}{h} u_0(U_{s,t}(x)) \int_0^h \Bigg[\nabla\varphi(U_{s,s+\tau}(x)) B(s+\tau, U_{s,s+\tau}(x)) \tilde{K}_{s,s+\tau}(x) \\
& - \varphi(U_{s,s+\tau}(x))(\delta_\sigma B_{s+\tau})(U_{s,s+\tau}(x)) \\
& \times \exp\left(- \int_t^{s+\tau} (\delta_\sigma B_r)(U_{s,r}(x))\, dr \right) \Bigg] \, d\tau \, d\sigma(x) \\
= & \int_0^h \int_{\mathbb{R}^d} \frac{1}{h} u_0(U_{s,t}(x)) [\nabla\varphi(U_{s,s+\tau}(x)) B(s+\tau, U_{s,s+\tau}(x)) \tilde{K}_{s,s+\tau}(x)] \, d\sigma(x) \, d\tau \\
& - \int_{\mathbb{R}^d} u_0(U_{s,t}(x)) \frac{1}{h} \int_0^h \Bigg[\varphi(U_{s,s+\tau}(x))(\delta_\sigma B_{s+\tau})(U_{s,s+\tau}(x)) \\
& \times \exp\left(- \int_s^{s+\tau} (\delta_\sigma B_r)(U_{s,r}(x))\, dr \right) \Bigg] \, d\tau \, d\sigma(x).
\end{aligned}
$$

The first integral converges to

$$\int_{\mathbb{R}^d} \nabla\varphi(x) u_0(U_{s,t}(x)) B(s,x) \, d\sigma(x)$$

when h goes to zero. The second integral converges to

$$\int_{\mathbb{R}^d} \varphi(x) u_0(U_{s,t}(x))(\delta_\sigma B_s)(x) \, d\sigma(x).$$

Using integration by parts, we obtain

$$\Delta_h(s) \to - \int_{\mathbb{R}^d} \varphi(x) v(s,x) \cdot \nabla B(s,x) \, d\sigma(x) \quad \text{when } h \to 0.$$

Using Lemma 3.3 and the arbitrariness of u_0 the result follows. $\square$

Remark (Concerning Theorem 1.2). (1) To prove Theorem 2.2, we begin with the regularization of the vector field made in (3.1). The assumptions of Theorem 2.2 imply all the assumptions of Theorem 2.1 except (3). This one was used just in (3.3) to assure the non-explosion of $U^n_{s,t}$ on the interval $[s,T]$. In the case of Theorem 2.2, the vector field B^n satisfies the assumptions of Theorem 3.1 in [C], and that theorem assures that the flow $U^n_{s,t}$ is defined for $s \leqslant t \leqslant T$. Apart for the non-explosion all the results of Theorem 2.2 are proved in the same way.

5. Avoidance of some sets in $\mathbb{R}^d$

In this paragraph we follow [A] to prove that the flow avoids some subsets in $\mathbb{R}^d$. The uniform integrability of the density functions is the fundamental ingredient.

Definition 1. Let A be a compact subset of $\mathbb{R}^d$. We denote $d_A(x) = \mathrm{dist}(x, A)$ and define

$$\lambda(A) = \sup\{\alpha : \sup_{0<z<1} z^{-\alpha}\sigma(\{d_A(x) < z\}) < \infty\}. \tag{5.1}$$

We have

Theorem 5.1. *Let B satisfy the assumptions of the Theorem* 2.1, *and A be a compact subset of $\mathbb{R}^d$ such that*

$$\frac{1}{q} + \frac{1}{\lambda(A)} < 1 \quad \textit{with } 1 < q^2 < 1 + \varepsilon. \tag{5.2}$$

Then the flow $U_{s,t}$ avoids the set A on the interval $s < t \leqslant T$.

Proof. Let us define

$$\tau_\delta(x) = \sup\{u \in [s, T] : d_A(U_{s,t}(x)) \geqslant \delta \quad \forall t \in [s, u]\}$$

if $d_A(x) > \delta$ and $\tau_\delta(x) = 0$ if $d_A(x) < \delta$.

For fixed t, r_0 such that $s < t < T$, $0 < r_0 < 1$ we consider $0 < \delta < r_0$, and we define the set

$$F_\delta = \{x : d_A(x) \geqslant r_0, \tau_\delta(x) \leqslant t\}.$$

For a.e. $x \in F_\delta$, $d_A(U_{s,\tau_\delta(x)}(x)) = \delta$. Taking the function

$$f(y) = \begin{cases} \log\left(\dfrac{r_0}{y}\right) & \text{if } 0 < y \leqslant r_0, \\ 0 & \text{if } y > r_0, \end{cases}$$

we have

$$\begin{aligned} \sigma(F_\delta)|f(\delta)| &= \int_{F_\delta} |f(d_A(U_{s,\tau_\delta(x)}(x))) - f(d_A(x))|\, d\sigma(x) \\ &\leqslant \int_{F_\delta} \int_s^{\tau_\delta(x)} |f'(d_A(U_{s,u}(x)))||B(u, U_{s,u}(x))|\, du\, d\sigma(x) \end{aligned}$$

$$
\begin{aligned}
&\leqslant \int_s^t \int_{\mathbb{R}^d} |f'(d_A(x))||B(u,x)|K_{s,u}(x)\,du\,d\sigma(x) \\
&= \int_s^t \int_{d_A(x)\leqslant r_0} \frac{1}{d_A(x)}|B(u,x)|K_{s,u}(x)\,du\,d\sigma(x) \\
&\leqslant \int_s^T \left(\int_{d_A(x)\leqslant r_0} \left(\frac{1}{d_A(x)}\right)^p d\sigma\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^d} |B(u,x)|^{q^2}\,d\sigma(x)\right)^{\frac{1}{q^2}} \\
&\quad \times \sup_{s<u\leqslant T} \|K_{s,u}\|_{L^{pq}(\sigma)},
\end{aligned}
$$

where $\frac{1}{q}+\frac{1}{p}=1$. Therefore

$$
\begin{aligned}
\sigma(F_\delta) \;\leqslant\; \left(\log\left(\frac{r_0}{\delta}\right)\right)^{-1} \left(\int_{d_A(x)\leqslant r_0} \left(\frac{1}{d_A(x)}\right)^p d\sigma\right)^{\frac{1}{p}} \\
\times \|B(u,x)\|_{L^{q^2}(\sigma)} \sup_{s<u\leqslant T} \|K_{s,u}\|_{L^{qp}(\sigma)}.
\end{aligned} \tag{5.3}
$$

Since $\lambda(A)$ defined in (5.1) satisfies (5.2), $p\leqslant\lambda(A)$.

Now, under the assumptions of the theorem,

$$
\left(\int_{d_A(x)\leqslant r_0} \left(\frac{1}{d_A(x)}\right)^p d\sigma\right)^{\frac{1}{p}}
$$

is finite. (cf. [A] for the proof of this property). Taking the limit in (5.3) as $\delta\to 0$, we deduce that $\sigma(F_\delta)\to 0$, which implies that $\sigma(\bigcap_{0<\delta<r_0} F_\delta)=0$. Since t and r_0 are arbitrary this proves the theorem. $\square$

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